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RESEARCH PAPER

On the Signed Complete Graphs with Maximum Index

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Abstract
Let $\Gamma = (K_n, H^-)$ be a signed complete graph whose negative edges induce a subgraph H. The index of Γ is the largest eigenvalue of its adjacency matrix. In this paper, we study the index of Γ when H is a unicyclic graph. We show that among all signed complete graphs of order $n > 5$ whose negative edges induce a unicyclic graph of order k and maximizes the index, the negative edges induce a triangle with all remaining vertices being pendant at the same vertex of the triangle.

Keywords Signed complete graph · Unicyclic graph · Index

Mathematics Subject Classification 05C22 · 05C50 · 15A18

1 Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G is defined as $|V(G)|$. The degree of a vertex v in G is denoted by $\deg_G(v)$. A vertex of degree one is called a *pendant vertex*. We denote the set of all neighbors of v in G by $N_G(v)$. Let K_n be the *complete graph* of order n, and $K_{1,k}$ denotes the *star graph* of order $k+1$. A tree containing exactly two non-pendant vertices is called a *double-star*. A double-star with degree sequence $(s+1, t+1, 1, \ldots, 1)$ is denoted by $D_{s,t}$. By C_k , we denote a cycle of length k. A *unicyclic* graph is a connected graph containing exactly one cycle. A *cactus* is a connected graph in which any two cycles have at most one common vertex.

A *signed graph* Γ is an ordered pair (G, σ), where G is a simple graph (called the *underlying graph*), and $\sigma : E(G) \longrightarrow \{-, +\}$ is a mapping defined on the edge set of G (called the *signature*). If all edges of a signed graph (G, σ) are positive (resp. negative), then we denote it by $(G, +)$ (resp. $(G, -)$). For a subset $X \subseteq V(\Gamma)$, the subgraph induced by X is denoted by $\Gamma[X]$. Let $A(G) = (a_{ij})$ be the adjacency matrix of G. The *adjacency matrix* of a signed graph $\Gamma = (G, \sigma)$ is defined as a square matrix $A(\Gamma) = (a^{\sigma}_{ij})$, where $a^{\sigma}_{ij} = \sigma(v_i v_j) a_{ij}$. The characteristic polynomial of a matrix A is denoted by $\varphi(A, \lambda)$. The characteristic polynomial of $A(\Gamma)$ is called the *characteristic polynomial* of the signed graph Γ and denoted by $\varphi(\Gamma, \lambda)$. Also, the spectrum of $A(\Gamma)$ is called the spectrum of Γ , and the largest eigenvalue is often called the *index*. The spectrum of signed graphs has been studied by many authors, for instance see Akbari et al. (2019), Belardo and Petecki (2015), Stanić (2019, 2018).

Let $\Gamma = (G, \sigma)$ be a signed graph and $v \in V(\Gamma)$. We obtain a new graph Γ' from Γ if we change the signs of all edges incident with v. We call v a *switching vertex*. A *switching* of a signed graph Γ is a graph that can be obtained by applying finitely many switching operations. We call two graphs Γ and Γ' *switching equivalent* if Γ' is a switching of Γ , and we write $\Gamma \sim \Gamma'$. The adjacency matrices of two switching equivalent signed graphs Γ and Γ' are similar, and hence, they have the same spectrum, see Zaslavsky (2008).

In Koledin and Stanić (2017), the connected signed graphs of fixed order, size, and number of negative edges with maximum index have been studied. It was conjectured in Koledin and Stanić (2017) that if Γ is a signed complete

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graph of order n with k negative edges, $k<n-1$ and Γ has maximum index, then negative edges induce the signed star $K_{1,k}$. In Akbari et al. (2020), the authors proved that this conjecture holds for signed complete graphs whose negative edges form a tree. Recently, Ghorbani and Majidi (2021) proved the conjecture. Let (K_n, H^-) denote a signed complete graph of order n whose negative edges induce a subgraph H. They introduced a family of graphs $H_{n,m}$ for positive integers n and m with $m \leq \lfloor \frac{n^2}{4} \rfloor$ and proved that among the signed complete graphs with n vertices and m negative edges, (K_n, H^-) has the maximum index if and only if H is isomorphic to a $H_{n,m}$.

In this paper, we focus on the signed complete graphs whose negative edges induce a unicyclic graph. We show that among all signed complete graphs of order $n > 5$ whose negative edges induce a unicyclic graph of order k and maximizes the index, the negative edges induce a triangle with all remaining vertices being pendant at the same vertex of the triangle. This result with a result of Akbari et al. (2020) on trees lead to a conjecture on signed complete graphs whose negative edges induce a cactus graph.

2 Preliminaries

The spectral theory of signed graphs has more varieties than unsigned graphs. But an important tool works in a similar way for signed graphs, which is a consequence of (Cvetković et al. 2010, Theorem 1.3.11).

Theorem 1 (Interlacing Theorem for signed graphs) Let Γ be a signed graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and let Γ' be an induced subgraph of Γ with m vertices. If the eigenvalues of Γ' are $\mu_1 \geq \cdots \geq \mu_m$, then $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ for $i = 1, \ldots, m$.

The following results will be useful in the sequel.

Lemma 1 (*Akbari et al.* 2020, *Lemma* 3) Let $\Gamma = (K_n, K_{1,k}^-)$ be a signed complete graph. Then

$$\varphi(\Gamma, \lambda) = (\lambda+1)^{n-3}\Big(\lambda^3 + (3-n)\lambda^2 + (3-2n)\lambda + 4k(n-k-1) + 1 - n\Big).$$

Theorem 2 (Akbari et al. 2020, Theorem 4 and Remark 5) Let $\Gamma = (K_n, D_{s,t}^-)$ be a signed complete graph and $n \geq s+t+2$. Then

$$\begin{aligned}\varphi(\Gamma, \lambda) =& (\lambda+1)^{n-5}\Big(\lambda^5 + (5-n)\lambda^4 + (10-4n)\lambda^3 \\ &+ (4ku + 8st + 10 - 6n)\lambda^2 \\ &+ (8ku + 16st + 5 - 4n)\lambda \\ &+ 4ku + 8st - 16stu + 1 - n\Big),\end{aligned}$$

where $k = s + t + 1$ and $u = n - s - t - 2$.

Also, we need to introduce an additional notation. Assume that A is a symmetric real matrix of order n, and $\{X_1, \ldots, X_m\}$ is a partition of $[n] = \{1, \ldots, n\}$. Let $\{X_1, \ldots, X_m\}$ partition the rows and columns of A, as follows,

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix},$$

where $A_{i,j}$ denotes the submatrix of A formed by rows in X_i and the columns in X_j. Then the $m \times m$ matrix $B = (b_{ij})$ is called the *quotient matrix* related to that partition, where b_{ij} denotes the average row sum of $A_{i,j}$. If the row sum of each $A_{i,j}$ is constant, then the partition is called *equitable*, see Brouwer and Haemers (2012).

If $X = (x_1, \ldots, x_n)^T$ is an eigenvector corresponding to the eigenvalue λ of a signed graph $\Gamma = (G, \sigma)$, then we assume that the component x_v corresponds to the vertex v. So, the following is the *eigenvalue equation* for v:

$$\lambda x_v = \sum_{u \in N_G(v)} \sigma(uv) x_u.$$

The next lemma is one of the most used tools in the identifications of graphs with maximum index.

Lemma 2 (*Koledin and Stanić 2017, Lemma* 5.1(*i*)) *Let u, v, and w be distinct vertices of a signed graph Γ and let $X = (x_1, \ldots, x_n)^T$ be an eigenvector corresponding to the index $\lambda_1(\Gamma)$. Let Γ' be obtained by reversing the sign of the positive edge uv and the negative edge uw. If $x_u \geq 0$, $x_v \leq x_w$ or $x_u \leq 0$, $x_v \geq x_w$, then $\lambda_1(\Gamma) \leq \lambda_1(\Gamma')$. If at least one inequality is strict, then $\lambda_1(\Gamma) < \lambda_1(\Gamma')$.*

If u, v, and w are the vertices given in Lemma 2, then $R(u, v, w)$ refers to the relocation described in the lemma.

3 (K_n, U^-) with maximum index

One classical problem of graph spectra is to identify the maximal graphs with respect to the index in a given class of graphs. In this section, we determine (K_n, U^-) with maximum index, where U is a unicyclic subgraph of K_n of order k. We begin with the following lemma.

Lemma 3 *Let $\Gamma = (K_n, Q_1^-)$ be a signed complete graph with $k \geq 4$ negative edges, where Q_1 is the graph depicted in Fig. 1. Then*

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-5}\Big(\lambda^5+(5-n)\lambda^4+(10-4n)\lambda^3\\ &+(12k-6n+4ku-38)\lambda^2\\ &+(24k-4n+8ku-91)\lambda\\ &+127n-116k-28ku-47\Big),\end{aligned}$$

where $u = n - k$.

Proof Let Q_1 be the graph depicted in Figure 1. First assume that $k<n$. By switching Γ at vertex v_1, one can deduce that $\Gamma = (K_n, Q_1^-)$ is switching equivalent to $(K_n, D^-_{n-k,2})$. Thus, the characteristic polynomials of $\Gamma = (K_n, Q_1^-)$ and $(K_n, D^-_{n-k,2})$ are the same. Hence by Theorem 2, the result holds. Now, assume that $k = n$, and so $u = 0$. Again by switching Γ at vertex v_1, we conclude that $\Gamma = (K_n, Q_1^-)$ is switching equivalent to $(K_n, K^-_{1,3})$. Therefore, by Lemma 1,

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-3}\Big(\lambda^3+(3-n)\lambda^2+(3-2n)\lambda\\ &+11n-47\Big)\\ =&(\lambda+1)^{n-5}\Big(\lambda^5+(5-n)\lambda^4+(10-4n)\lambda^3\\ &+(6n-38)\lambda^2+(20n-91)\lambda+11n-47\Big).\end{aligned}$$

Hence, the proof is complete. □

Let $\Gamma = (K_n, \sigma)$ be a signed complete graph. Let $\sqcap = \{X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}\}$ be a partition of $V(\Gamma)$ such that all edges between X_i and X_j have the same sign for each i, j, $\Gamma[X_i] = (K_{n_i}, +)$ for $i = 1, \ldots, p$, and $\Gamma[X_i] = (K_{n_i}, -)$ for $i = p+1, \ldots, p+q$, where $|X_i| = n_i$ for $i = 1, \ldots, p+q$. Obviously, $\sqcap$ is an equitable partition of $V(\Gamma)$. We call $\sqcap$ a *special partition* of $V(\Gamma)$. Moreover, we have the next theorem.

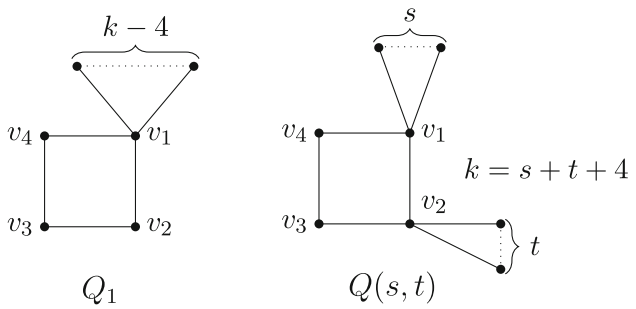


Fig. 1 The unicyclic graphs Q_1, $Q(s, t)$

Theorem 3 *Let $\Gamma = (K_n, \sigma)$ be a signed complete graph. Let $\sqcap = \{X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}\}$ be a special partition of $V(\Gamma)$, and B be the quotient matrix of $A(\Gamma)$ related to $\sqcap$. If $m_1 = \sum_{i=1}^{p} n_i$ and $m_2 = \sum_{i=p+1}^{p+q} n_i$, then*

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{m_1-p}(\lambda-1)^{m_2-q}\varphi(B,\lambda).$$

Proof Suppose that $X_1 = \{v_1, \ldots, v_{n_1}\}$ and $X_i = \{v_{n_1+\cdots+n_{i-1}}+1, \ldots, v_{n_1+\cdots+n_i}\}$, for $i = 2, \ldots, p+q$. Then we have,

$$\lambda I - A(K_n,\sigma) = \left[\begin{array}{c|c} \lambda I - A(K_{n_1},+) & * \\ \hline * & * \end{array}\right].$$

We apply finitely many elementary row and column operations on the matrix $\lambda I - A(K_n, \sigma)$. First, subtract the n_1th row from all the upper rows. This leads to the following matrix,

$$\left[\begin{array}{cccc|c} \lambda+1 & & & -\lambda-1 & \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots & \mathbf{0} \\ & & \lambda+1 & -\lambda-1 & \\ & \mathbf{-1} & & \lambda & * \\ \hline & & * & & * \end{array}\right].$$

Now, adding the first $n_1 - 1$ columns to n_1th column, we obtain the following matrix,

$$\left[\begin{array}{cccc|c} \lambda+1 & & & 0 & \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots & \mathbf{0} \\ & & \lambda+1 & 0 & \\ & \mathbf{-1} & & \lambda+1-n_1 & * \\ \hline & & * & & * \end{array}\right].$$

Thus, the following holds:

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{n_1-1}\det\left[\begin{array}{c|c} \lambda+1-n_1 & * \\ \hline * & * \end{array}\right].$$

By repeating the same procedure, one can deduce that

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{m_1-p}(\lambda-1)^{m_2-q}\varphi(B,\lambda),$$

as desired. □

Lemma 4 *Let $\Gamma = (K_n, Q(s,t)^-)$, $s, t \geq 1$, be a signed complete graph with k negative edges, where Q(s, t) is the graph depicted in Fig. 1. Then*

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-7}\Big(\lambda^7+(7-n)\lambda^6+(21-6n)\lambda^5\\&+(12k-15n+4ku+8st-13)\lambda^4\\&+(48k-20n+16ku+32st-157)\lambda^3\\&+(113n-56k-8ku-16st(u-1)-267)\lambda^2\\&+(250n-208k-48ku-32st(u+1)-185)\lambda\\&+127n-116k-28ku+24st(2u-1)-47\Big),\end{aligned}$$

where $u=n-k$.

Proof Assume that $k<n$. Suppose that $X_1=\{v_1\}$, $X_2=\{v_2\}$, $X_3=\{v_3\}$, $X_4=\{v_4\}$, $X_5=N_{Q(s,t)}(v_1)\setminus\{v_2,v_4\}$, $X_6=N_{Q(s,t)}(v_2)\setminus\{v_1,v_3\}$, and $X_7=V(K_n)\setminus V(Q(s,t))$, see Fig. 1. Let B be the quotient matrix of $A(\Gamma)$ related to $\{X_1,\ldots,X_7\}$. It is not hard to see that the characteristic polynomial of B is

$$\begin{aligned}\varphi(B,\lambda) =&\lambda^7+(7-n)\lambda^6+(21-6n)\lambda^5\\&+(12k-15n+4ku+8st-13)\lambda^4+\\&(48k-20n+16ku+32st-157)\lambda^3\\&+(113n-56k-8ku-16st(u-1)-267)\lambda^2\\&+(250n-208k-48ku-32st(u+1)-185)\lambda\\&+127n-116k-28ku+24st(2u-1)-47,\end{aligned}$$

where $u=n-k$. By Theorem 3, we have $\varphi(\Gamma,\lambda)=(\lambda+1)^{n-7}\varphi(B,\lambda)$. Now, assume that $k=n$. Let $X_1,\ldots,X_6$ be as above, and B' be the quotient matrix of $A(\Gamma)$ related to $\{X_1,\ldots,X_6\}$. Then $\varphi(\Gamma,\lambda)=(\lambda+1)^{n-6}\varphi(B',\lambda)$. So, one can deduce that

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-6}\Big(\lambda^6+(6-n)\lambda^5+(15-5n)\lambda^4\\&+(2n+8st-28)\lambda^3+\\&(26n+24st-129)\lambda^2+(31n-8st-138)\lambda\\&+11n-24st-47\Big),\end{aligned}$$

and hence

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-7}\Big(\lambda^7+(7-n)\lambda^6+(21-6n)\lambda^5\\&+(-3n+8st-13)\lambda^4\\&+(28n+32st-157)\lambda^3+(57n+16st-267)\lambda^2\\&+(42n-32st-185)\lambda+11n-24st-47\Big).\end{aligned}$$

This completes the proof. □

Corollary 1 *Let $\Gamma=(K_n,Q(s,t)^-)$, $s,t\geq 1$, and $\Gamma'=(K_n,Q_1^-)$ be two signed complete graphs with k negative edges, where $Q(s,t)$ and Q_1 are the graphs depicted in Fig. 1. Then $\lambda_1(\Gamma)<\lambda_1(\Gamma')$.*

Proof Let $\lambda_1=\lambda_1(\Gamma)$. By Lemmas 3 and 4, we deduce that

$$\begin{aligned}\varphi(\Gamma',\lambda)-\varphi(\Gamma,\lambda) =&-8st(\lambda+1)^{n-7}\big(\lambda^4\\&+4\lambda^3-2(n-k-1)\lambda^2-4(n-k+1)\lambda\\&+3(2n-2k-1)\big).\end{aligned}$$

Setting $\lambda=\lambda_1$, we have

$$\begin{aligned}\varphi(\Gamma',\lambda_1) =&-8st(\lambda_1+1)^{n-7}\big(\lambda_1^4+4\lambda_1^3\\&-2(n-k-1)\lambda_1^2-4(n-k+1)\lambda_1\\&+3(2n-2k-1)\big).\end{aligned}$$

Since Γ has $(K_{n-3},+)$ as an induced subgraph, by Theorem 1, we conclude that $n-k+1<n-4\leq\lambda_1$. Therefore, $\varphi(\Gamma',\lambda_1(\Gamma))<0$ which implies that $\lambda_1(\Gamma)<\lambda_1(\Gamma')$. □

By a proof similar to the proof of Lemma 4, we have the following lemma.

Lemma 5 *Let $\Gamma=(K_n,U_1^-)$ be a signed complete graph with $k\geq 3$ negative edges, where U_1 is the graph depicted in Fig. 2. Then*

$$\begin{aligned}\varphi(\Gamma,\lambda) =&(\lambda+1)^{n-5}(\lambda-1)\Big(\lambda^4+(6-n)\lambda^3\\&+(16-5n)\lambda^2+(4k-11n+4ku+18)\lambda\\&+28k-31n+12ku+7\Big),\end{aligned}$$

where $u=n-k$.

Now, we can state the main result of this paper.

Theorem 4 *Among all signed complete graphs of order $n>5$ whose negative edges induce a unicyclic graph of order k and maximizes the index, the negative edges induce a triangle with all remaining vertices being pendant at the same vertex of the triangle.*

Proof Suppose that k negative edges induce a unicyclic graph U and maximizes the index, where $k\leq n$. Let $\Gamma=(K_n,U^-)$ and $\lambda_1=\lambda_1(\Gamma)$. Clearly, $(K_{n-k+1},+)$ is an induced subgraph of Γ. Hence by Theorem 1, we deduce that $n-k\leq\lambda_1$. Suppose U consists of the cycle, say C, of length g and a certain number of trees attached at vertices

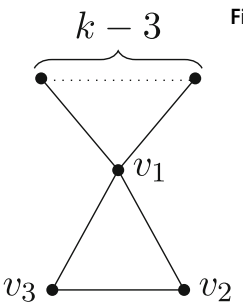


Fig. 2 The unicyclic graph U_1

of C. Let $V(U)=\{v_1,\ldots,v_k\}$ and $V(C)=\{v_1,\ldots,v_g\}$. Let $X=(x_1,\ldots,x_n)^T$ be an eigenvector corresponding to λ_1.

We claim that there is an integer i, $1\le i\le g$, such that $x_i\neq 0$. To see this, by contrary assume that $x_i=0$, for $i=1,\ldots,g$. Let v_p and v_q be two consecutive vertices of C, that is $v_pv_q\in E(C)$, and suppose that two vertices v_j $(g<j\le k)$ and v_p are adjacent in U. If $x_j\neq 0$, then by Lemma 2, the relocation $R(v_j,v_q,v_p)$ would contradict the maximality of λ_1. By the same argument, one can prove that $x_i=0$, for $i=1,\ldots,k$. Thus $k<n$. Let x_t be an entry of X having the largest absolute value. We may assume that $x_t>0$ (otherwise, consider $-X$ instead of X). By the eigenvalue equation for v_t, we have $\sum_{i=k+1,i\neq t}^{n}x_i=\lambda_1x_t$ which implies that $\lambda_1\le n-k-1$, a contradiction. The claim is proved.

Suppose that $g>3$. Without loss of generality, assume that $x_1>0$. If $x_{g-1}\le x_2$, then the possibility of $R(v_1,v_{g-1},v_2)$ contradicts the maximality of λ_1. So $x_2<x_{g-1}$. Also, $x_g<x_3$ since otherwise, the relocation $R(v_1,v_3,v_g)$ leads to a contradiction. Now, if $x_2\ge 0$ (resp. $x_g\ge 0$), then the relocation $R(v_2,v_g,v_3)$ (resp. $R(v_g,v_2,v_{g-1})$) contradicts the maximality of λ_1. Hence, $x_2,x_g<0$. Thus for $g\ge 5$, if $x_{g-1}\ge x_3$ (resp. $x_3\ge x_{g-1}$), then by $R(v_2,v_{g-1},v_3)$ (resp. $R(v_g,v_3,v_{g-1})$), we get a contradiction. Therefore, $g=4$.

From the previous statement, we have $x_1>0$ and $x_2,x_4<0$. Also, $x_3>0$ since otherwise, the relocation $R(v_3,v_1,v_2)$ concludes a contradiction. Suppose that $v_1v_p,v_3v_q\in E(U)$, where $p,q>4$. If $x_p\ge 0$ (resp. $x_q\ge 0$), then the relocation $R(v_p,v_2,v_1)$ (resp. $R(v_q,v_4,v_3)$) contradicts the maximality of λ_1. Thus $x_p,x_q<0$. So if $x_3\ge x_1$ (resp. $x_1\ge x_3$), then by $R(v_p,v_3,v_1)$ (resp. $R(v_q,v_1,v_3)$), we find a contradiction. It follows that $\deg_U(v_1)=2$ or $\deg_U(v_3)=2$. Similarly, if $v_2v_p,v_4v_q\in E(U)$ and $p,q>4$, then we have a contradiction and hence $\deg_U(v_2)=2$ or $\deg_U(v_4)=2$. Without loss of generality, assume that $\deg_U(v_3)=\deg_U(v_4)=2$.

Now, we prove that the trees attached to vertices v_1,v_2 in U, if any, are stars. To prove this, first assume that $v_1v_p,v_pv_q\in E(U)$, where $p,q>4$. If $x_p\ge 0$, then the relocation $R(v_p,v_2,v_1)$ gives a contradiction and hence $x_p<0$. If $x_q\le 0$, then the relocation $R(v_q,v_1,v_p)$ leads to a contradiction. Thus $x_q>0$. Therefore, if $x_2\le x_p$ (resp. $x_p\le x_2$), then by $R(v_q,v_2,v_p)$ (resp. $R(v_3,v_p,v_2)$), we get a contradiction. This implies that v_p is a pendant vertex of U. By repeating the same procedure, one can easily prove that the tree which is attached to the vertex v_2 in U, if any, is a star. So, we conclude that U may be equal to the graphs Q_1 or $Q(s,t)$, depicted in Fig. 1. By Corollary 1, $\lambda_1(K_n,Q(s,t)^-)<\lambda_1(K_n,Q_1^-)$ and hence $U\neq Q(s,t)$.

Next, suppose that $g=3$. We first prove that at least two vertices of the cycle C have degree 2 in U. By contrary assume that $v_1v_p,v_2v_q\in E(U)$, where $p,q>3$. If $x_1=x_2=0$, then $R(v_p,v_2,v_1)$ implies that $x_p=0$. Hence by $R(v_3,v_p,v_1)$, we deduce that $x_3=0$, a contradiction. Without loss of generality, assume that $x_1>0$. If $x_q\le x_2$ (resp. $x_q\le x_3$), then the relocation $R(v_1,v_q,v_2)$ (resp. $R(v_1,v_q,v_3)$) contradicts the maximality of λ_1. Thus $x_q>x_2,x_3$. Therefore, $x_3>0$ (by $R(v_3,v_q,v_2)$). If $x_1\le x_2$, then $R(v_q,v_1,v_2)$ yields a contradiction. So $x_2<x_1$ and hence $x_p<0$ (by $R(v_p,v_2,v_1)$). Now, $R(v_3,v_p,v_1)$ gives the final contradiction. This completes the assertion.

We now prove that the tree attached to a vertex of C in U, if any, is a star. By contrary assume that $v_1v_p,v_pv_q\in E(U)$, where $p,q>3$. Without loss of generality, assume that $x_q\ge 0$ (otherwise, consider $-X$ instead of X). If $x_1<x_p$, then the relocation $R(v_q,v_1,v_p)$ contradicts the maximality of λ_1. Hence $x_p\le x_1$. Thus the relocations $R(v_2,v_p,v_1)$ and $R(v_3,v_p,v_1)$, respectively, imply that $x_2\le 0$ and $x_3\le 0$. Therefore, by $R(v_2,v_q,v_3)$, we deduce that $x_2=x_3=x_q=0$. Finally, the relocation $R(v_1,v_q,v_2)$ yields $x_1=0$ which is impossible, and we are done.

So the candidates for maximizers are: $\Gamma=(K_n,U_1^-)$, and $\Gamma'=(K_n,Q_1^-)$, see Figs. 1 and 2. We just have to compare the indices. By Lemmas 3 and 5, we find that

$$\varphi(\Gamma,\lambda)-\varphi(\Gamma',\lambda)=-8(\lambda+1)^{n-5}\big((k-5)\lambda^2+2(n-5)\lambda+12n-11k-2k(n-k)-5\big).$$

Let $\lambda_1=\lambda_1(\Gamma')$. Since Γ' has $(K_{n-2},+)$ as an induced subgraph, by Theorem 1, we conclude that $\lambda_1\ge n-3$. Setting $\lambda=\lambda_1$, we have

$$\varphi(\Gamma,\lambda_1)=-8(\lambda_1+1)^{n-5}\big((k-5)\lambda_1^2+2(n-5)\lambda_1+12n-11k-2k(n-k)-5\big).$$

First assume that $k\ge 5$. Thus $(k-5)\lambda_1^2+2(n-5)\lambda_1+12n\ge(k-5)(n-3)^2+2(n-5)(n-3)+12n=kn^2+26n+9k-3n^2-6kn-15$. It is not hard to see that $kn^2+2k^2+26n>3n^2+8kn+2k+20$ which implies that $kn^2+26n+9k-3n^2-6kn-15>11k+2k(n-k)+5$. Hence $\varphi(\Gamma,\lambda_1)<0$ and so $\lambda_1(\Gamma')<\lambda_1(\Gamma)$. Note that if $n=k=5$, by switching Γ' at vertex v_3, we have $\Gamma'\sim\Gamma$ and hence $\lambda_1(\Gamma')=\lambda_1(\Gamma)$. Finally, assume that $k=4$. Thus

$$\varphi(\Gamma,\lambda_1)=8(\lambda_1+1)^{n-5}\big(\lambda_1^2-2(n-5)\lambda_1-4n+17\big).$$

The roots of $\lambda^2-2(n-5)\lambda-4n+17$ are $n-5\pm\sqrt{n^2-6n+8}$. If $n>7$, then $n-5-\sqrt{n^2-6n+8}<n-3\le\lambda_1\le n-1<n-5+\sqrt{n^2-6n+8}$, so $\varphi(\Gamma,\lambda_1)<0$ which yields that $\lambda_1(\Gamma')<\lambda_1(\Gamma)$. By a

computer search, one can see that the result holds for $n = 6, 7$, but $\lambda_1(K_4, U_1^-) < \lambda_1(K_4, C_4^-)$ and $\lambda_1(K_5, U_1^-) < \lambda_1(K_5, C_4^-)$. The proof is complete. □

The following is a direct consequence of Theorem 4 which confirms the Koledin-Stanić conjecture for signed complete graphs whose negative edges induce a unicyclic graph.

Corollary 2 *Let (K_n, U^-) be a signed complete graph whose negative edges induce a unicyclic graph U of order $k < n$. Then $\lambda_1(K_n, U^-) < \lambda_1(K_n, K_{1,k}^-)$.*

Proof Suppose that $\Gamma = (K_n, K_{1,k}^-)$ and $\Gamma' = (K_n, U_1^-)$, where U_1 is the graph depicted in Fig. 2. Let $\lambda_1 = \lambda_1(\Gamma')$. Since Γ' has $(K_{n-2}, +)$ as an induced subgraph, by Theorem 1, we conclude that $\lambda_1 \geq n - 3$. By Lemmas 1 and 5, we deduce that

$$\begin{aligned}\varphi(\Gamma, \lambda) - \varphi(\Gamma', \lambda) = &-8(\lambda+1)^{n-5}\big((k-1)\lambda^2 \\ &- 2(n-2k+1)\lambda \\ &+ 4n - 3k - 2k(n-k) - 1\big).\end{aligned}$$

Setting $\lambda = \lambda_1$, we have

$$\begin{aligned}\varphi(\Gamma, \lambda_1) = &-8(\lambda_1+1)^{n-5}\big((k-1)\lambda_1^2 \\ &- 2(n-2k+1)\lambda_1 \\ &+ 4n - 3k - 2k(n-k) - 1\big).\end{aligned}$$

If $k = 3$, then $\varphi(\Gamma, \lambda_1) = -16(\lambda_1+1)^{n-4}\big(\lambda_1 - n + 4\big)$. Thus $\varphi(\Gamma, \lambda_1) < 0$ and hence $\lambda_1(\Gamma') < \lambda_1(\Gamma)$. Note that $n \geq 4$.

Now, assume that $k > 3$. By a computer search, one can see that $\lambda_1(K_5, U_1^-) < \lambda_1(K_5, C_4^-) < \lambda_1(K_5, K_{1,4}^-)$, so the result holds for $n = 5$. In what follows, we consider two cases.

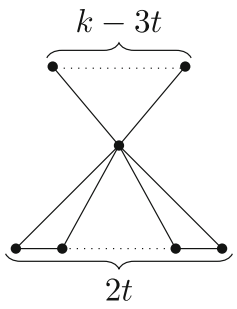
Case 1: $n > 2k - 1$. We claim that $\varphi(\Gamma, \lambda_1) < 0$. We have $(k-1)\lambda_1^2 + 4n \geq (k-1)(n-3)^2 + 4n = kn^2 + 10n + 9k - n^2 - 6kn - 9$. Since $\lambda_1 \leq n - 1$, hence

$$\begin{aligned}&2(n-2k+1)\lambda_1 + 3k + 2k(n-k) \\ &+ 1 \leq 2(n-2k+1)(n-1) + 3k + 2k(n-k) + 1 \\ &= 2n^2 + 7k - 2k^2 - 2kn - 1.\end{aligned}$$

It is easy to see that $kn^2 + 2k^2 + 10n + 2k > 3n^2 + 4kn + 8$ which yields that $kn^2 + 10n + 9k - n^2 - 6kn - 9 > 2n^2 + 7k - 2k^2 - 2kn - 1$. Therefore, $\lambda_1(\Gamma') < \lambda_1(\Gamma)$.

Case 2: $n \leq 2k - 1$. Then $(k-1)\lambda_1^2 - 2(n-2k+1)\lambda_1 + 4n \geq (k-1)(n-3)^2 - 2(n-2k+1)(n-3) + 4n$. It is not hard to check that $kn^2 + 2k^2 + 14n > 3n^2 + 6k + 4kn + 4$ which implies that $(k-1)(n-3)^2 - 2(n-2k+1)(n-3) + 4n > 3k + 2k(n-k) + 1$. Thus $\varphi(\Gamma, \lambda_1) < 0$ and hence $\lambda_1(\Gamma') < \lambda_1(\Gamma)$.

Fig. 3 The cactus graph G_t



Therefore, by Theorem 4, the proof is complete. □

Let G_t be the cactus graph with k edges and t cycles, depicted in Fig. 3 (G_0 is the star graph $K_{1,k}$). We close this paper with the following conjecture.

Conjecture 1 *Among all signed complete graphs of order $n > 5$ whose negative edges induce a cactus graph with k edges and t cycles $(k - t < n)$, and maximizes the index, negative edges induce the graph G_t.*

We note that Conjecture 1 is true for $t = 0, 1$, according to Theorem 4 and a result of Akbari et al. (2020).

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